Mathematical Induction

College of Computing & Information Technology
King Abdulaziz University

CPCS-222 – Discrete Structures

Outline

- Summations
  - Basically reviewing basics of summations
  - And how they tie into some induction problems
- Mathematical Induction
  - This is the focus of today’s lecture.

Summations

- Definition:
  - In very basic terms, a summation is the addition of a set of numbers.
- Example:
  - Let's say we want to sum the integers from 1 to 5

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

  - Here, the "i" is just a variable.
  - Let's look at this notation in more detail…

Summations

- A summation: So what is f(j)?

$$\sum_{j=m}^{n} f(j) \quad \text{or} \quad \sum\nolimits_{j=m}^{n} f(j)$$

- Here, f(j) is simply a function in terms of j
  - Just like f(x) = 2x+1 is a function in terms of x.
  - f(j) is simply some function in terms of j.

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
    sum += f(j);
```

Summations

- Example con’d:
  - Since “i” is just a variable, we can use any variable name…

$$\sum_{Jason=1}^{5} Jason = 1 + 2 + 3 + 4 + 5 = 15$$

  - We also recognize that a summation is merely summing (adding) the values of some given function
    - This far, we’ve only looked at this most simply function:
      - f(i) = i
      - And then we summed up those i terms.

Summations

- Example 2:
  - Now, let us choose our function to be i^2, and let us again sum this from 1 to 5.

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

    - So, the deal here is we are summing whatever the function is, and we are summing this from the lower limit all the way to the upper limit, adding all the values together
  - What if we let the function be 2i+1

$$\sum_{i=1}^{5} 2i + 1 = 3 + 5 + 7 + 9 + 11 = 35$$

Summations

- More summations:
  - Now let us write this purely in "function" form so we all see what is going on.

$$\sum_{i=1}^{5} f(i) = f(1) + f(2) + f(3) + f(4) + f(5)$$

  - Again, on the previous example, we let f(i) = 2i+1
  - Thus, we had…

$$\sum_{i=1}^{5} 2i+1 = (2(1)+1)+(2(2)+1)+(2(3)+1)+(2(4)+1)+(2(5)+1)$$

$$= 3+5+7+9+11 = 35$$

Evaluating Summations

- Example (Question 13, section 2.4):

$$\sum_{k=1}^{5}(k+1)$$

- 2 + 3 + 4 + 5 + 6 = 20

$$\sum_{j=0}^{4}(-2)^{j}$$

- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$$\sum_{i=1}^{10}3$$

- 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30

$$\sum_{j=0}^{8}\left(2^{j+1}-2^{j}\right)$$

- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^{10}-2^9) = 511$
  - Note that each term (except the first and last) is cancelled by another term

Summations

- Summing large amounts of numbers:
  - What if you were told to simply sum the numbers from 1 to 1,000.
    - There is no function to plug-n-chug into since we are simply adding the numbers together…so that part is easy
    - BUT, we are adding ONE THOUSAND numbers
    - This would take SOOOOOOOOOOOOOOOO long.

$$\sum_{i=1}^{1,000} i = 1 + 2 + 3 + \ldots + 672 + 673 + \ldots + 998 + 999 + 1000$$

    - Who wants to grab a calculator and give it a shot?
    - That would be crazy!
      - Thankfully, mathematical solutions to some of these common summations have been developed and proven

Summations

That's cool right.?.
Cuz it saves you a BUNCH of time!

- Summing large amounts of numbers:
  - What if you were told to simply sum the numbers from 1 to 1,000.

$$\sum_{i=1}^{1,000} i = 1+2+3+\ldots+672+673+\ldots+998+999+1000$$

    - We have what is called a "closed form solution" to this problem:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

***Here, n is simply a variable representing the upper limit.

    - So we simply plug in the 1000 for the n and we get our answer:

$$\sum_{i=1}^{1,000} i = \frac{1000(1000+1)}{2} = 500{,}500$$

Summations

- Mathematical Induction is a proof method that is used to proof a variety of things…including proving some of these closed form solutions such as the one just shown.
- But before we move on to that, one more thing on summations
- This following “**<u>thing</u>**” is VERY important towards understanding induction.
  - Later in the slides, I will refer back to this as “**<u>the thing</u>**”.

Summations

- Look back at the very first summation example:

$$\sum_{i=1}^{5} i = 1+2+3+4+5 = 15$$

- Now, look at this summation:

$$\sum_{i=1}^{6} i = \underbrace{1+2+3+4+5}_{\sum_{i=1}^{5} i} + 6 = 21$$

$$\sum_{i=1}^{5} i + 6 = 21$$

What is the same between these two summations?

  - We realize that this summation is THE SAME as the previous summation except that we are adding one additional number in the sequence

Summations

- Similarly, look at it in "function" form:

$$\sum_{i=1}^{5} f(i) = f(1) + f(2) + f(3) + f(4) + f(5)$$

- Now, look at this summation:

$$\sum_{i=1}^{6} f(i) = \underbrace{f(1) + f(2) + f(3) + f(4) + f(5)}_{\sum_{i=1}^{5} f(i)} + f(6)$$

  - Again, we realize that the second summation is THE SAME as the first summation except that we are adding one additional term in the sequence

Summations

- So now, remember this summation:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$

- Hopefully you can now recognize the following:

$$\sum_{i=1}^{n+1} i = \underbrace{1 + 2 + 3 + \ldots + (n-2) + (n-1) + n}_{\sum_{i=1}^{n} i} + (n+1)$$

$$\sum_{i=1}^{n} i \qquad + (n+1)$$

  - Again, we realize that the second summation is THE SAME as the first summation except that we are adding one additional term in the sequence

Summations

So, BLAH is the actual answer to this summation.

- So now let us do a make believe problem.
- Problem:
  - Given that $\sum_{i=1}^{n} f(i) = BLAH$
  - Solve this: $\sum_{i=1}^{n+1} f(i)$
  - We know that $\sum_{i=1}^{n+1} f(i) = \sum_{i=1}^{n} f(i) + f(n+1)$

$$= BLAH + f(n+1)$$

BLAH is the “accepted” value of this function when we sum it from i to n.

using our knowledge from previous slides…

- So we were able to solve the summation to the n+1 term <u>by using</u> the **<u>known answer</u>** to the n^{th} term

An aside…

Is spelling really that important?

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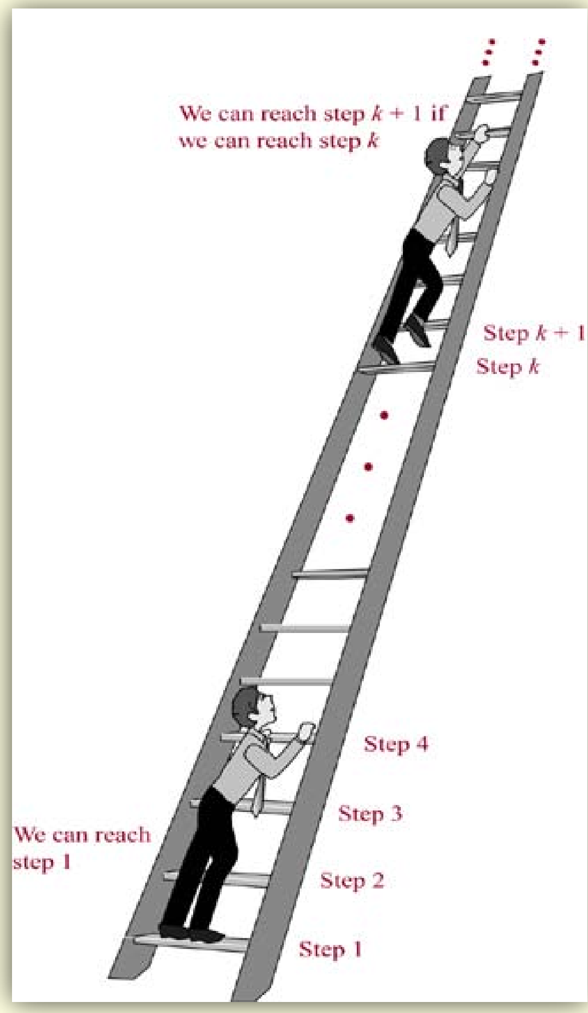
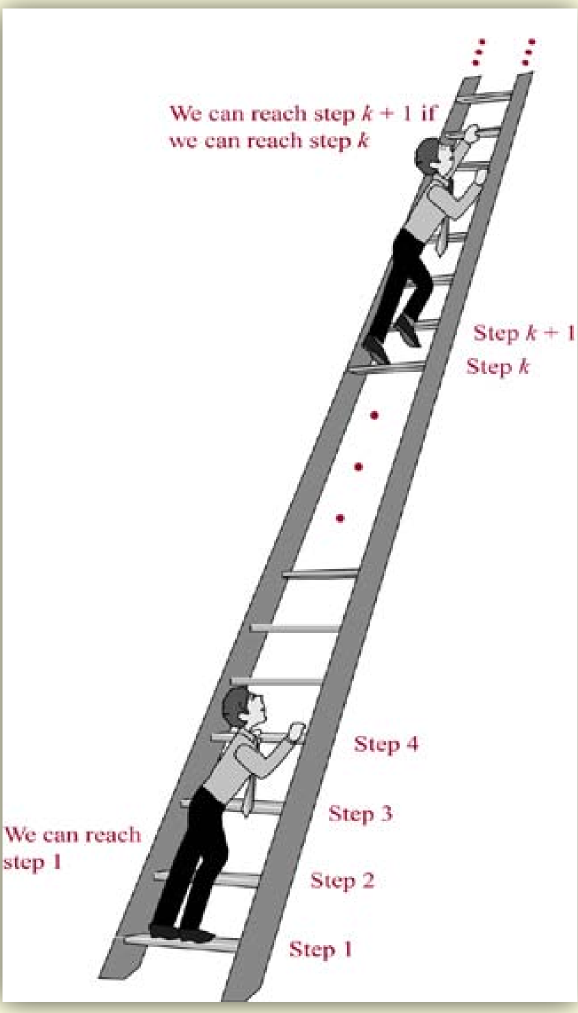
Mathematical Induction

- **Example: Infinite Ladder**
- Suppose we have an infinite ladder as show to the right.
- We want to know if we can reach every step on this ladder.
- We know two things (two rules):
  1) We can reach the first step of the ladder.
  2) And if we can reach a particular step of the ladder, then surely we can reach the next step after that.
- From this, can we conclude that we can reach every step?



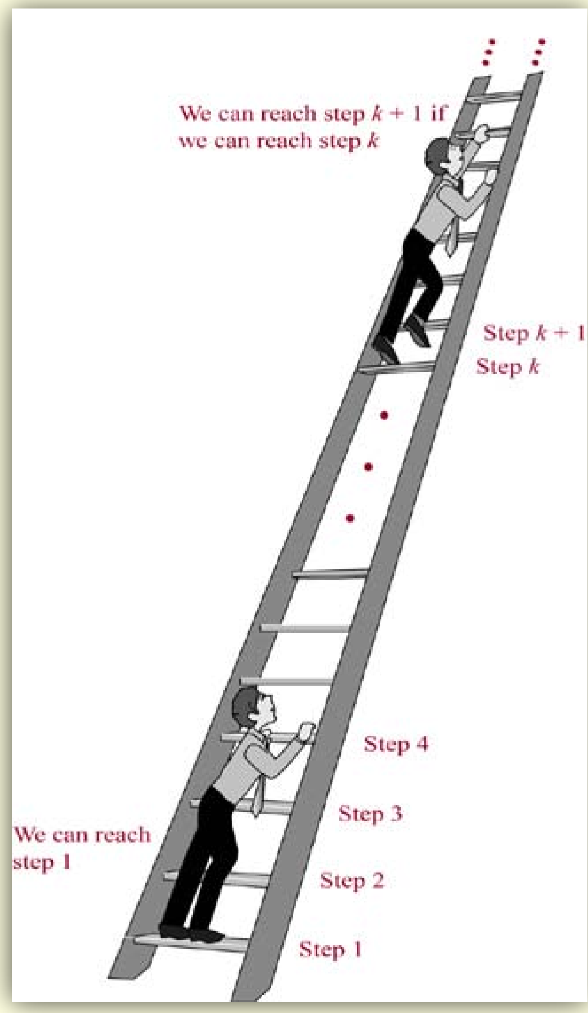
Mathematical Induction

- **Example: Infinite Ladder**
- Can we can reach every step?
  - We know we can reach step #1 (from rule 1 on previous slide).
  - Based on rule 2 (previous slide), we can then reach step #2.
  - Once we are at step #2, we can then reach step #3 (rule 2).
  - Continuing in this way, we can see that we can reach the 4th step, the 5th step, and so on (rule 2).
  - After 100 uses of rule 2, we know we can reach the 101st step.



Mathematical Induction

- **Example: Infinite Ladder**
- But can we conclude (prove) that we can reach each and every step on this infinite ladder?
  - The answer is YES!
  - We can prove this using Mathematical Induction
  - We let P(n) be the statement that we can reach the nth step of the ladder.
  - We can then show that P(n) is true for every positive integer n.
  - This is the basic concept of Induction.



What is induction?

- A method of proof
- It does not generate answers: it only can prove them
- Three parts:
  - Base case(s): show it is true for one element
  - Inductive hypothesis: **assume** it is true for an arbitrary positive integer k
    - **Must be clearly labeled!!!**
  - Inductive Step:

    Show it is true for k+1 based on the truth of k

Mathematical Induction

- Principle of Mathematical Induction:
  - Given some propositional function P(n), we prove that P(n) is true for all positive integers by completing three steps:
  - Base Step:
    - We verify that P(1) is true.
  - Inductive <u>Hypothesis</u>:
    - **<u>Assume</u>** that P(k) is true for an arbitrary integer k
  - Inductive Step:
    - From the assumption in the I.H., we show that P(k+1) must also be true.

Mathematical Induction

- Classical Example:
- We stated earlier that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- **Prove** this via Mathematical Induction.
  1) **<u>Base Case:</u>** We want to confirm that the left side (LS) equals the right side (RS) when we plug in the smallest value (which in this case is 1).
     - LS: $\sum_{i=1}^{1} i = 1$ RS: $\frac{1(1+1)}{2} = \frac{2}{2} = 1$
  2) **<u>Induction Hypothesis:</u>** assume true for an arbitrary k

     $$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \quad \text{for } k \geq 1$$

     <u>Assume</u> P(k) is True

Mathematical Induction

- Classical Example: Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by Induction

3) **<u>Induction Step:</u>** based on the I.H., prove P(k+1).

****At this step, you basically plug in a k+1 wherever you see a k (or n).*

$$\text{Prove } \sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2}$$

PROVE This!

Here's where we use "**<u>the thing</u>**" shown earlier.

$$\sum_{i=1}^{k+1} i = \underbrace{1+2+\ldots+(k-1)+k}+(k+1)$$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i \quad +(k+1)$$

Guess what:
From our Induction Hypothesis,
we know that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

So we substitute this in…

Mathematical Induction

- Classical Example: Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by Induction

3) **<u>Induction Step:</u>** based on the I.H., prove P(k+1).

$$\text{Prove } \sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2}$$

Here's where we use "the thing" shown earlier.

$$\sum_{i=1}^{k+1} i = \underbrace{1+2+\ldots+(k-1)+k}+(k+1)$$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

Guess what:
From our Induction Hypothesis, we know that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

$$\boxed{\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}}$$

Mathematical Induction

- Classical Example: Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by Induction

3) **<u>Induction Step:</u>** based on the I.H., prove P(k+1).

$$\text{Prove } \sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2}$$

- We just proved the Induction Step above. We proved that P(k+1) is true under the assumption that P(k) is true.
- Since we completed all three steps, by Mathematical Induction, we know that P(n) is true for all positive integers n.
- That is, we have proven, for all positive n, that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

What did we show

- Base case: P(1)
- If P(*k*) was true, then P(*k*+1) is true
  - i.e., P(*k*) → P(*k*+1)

- We know it's true for P(1)
- Because of P(*k*) → P(*k*+1), if it's true for P(1), then it's true for P(2)
- Because of P(*k*) → P(*k*+1), if it's true for P(2), then it's true for P(3)
- Because of P(*k*) → P(*k*+1), if it's true for P(3), then it's true for P(4)
- Because of P(*k*) → P(*k*+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it is true for all possible values of *n*

- In other words, we showed that:

$$\left[\mathrm{P}(1) \wedge \forall k\left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)\right] \rightarrow \forall n\, \mathrm{P}(n)$$

The idea behind Inductive Proofs

- Show the base case
- Show the inductive hypothesis
- **<u>Manipulate</u>** the inductive step so that you can substitute in part of the inductive hypothesis
  - Use rules of math to manipulate the equation of the inductive step
  - The goal: you want to “see” the left-hand-side of the inductive hypothesis within the equation of the inductive step
  - Then you can substitute the right-hand-side into it
- Show the inductive step

Mathematical Induction

- Example #2:
  - Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.
  - So firstly, we need to figure out the formula that calculates the sum of the first n positive odd integers
    - The sums of the first n positive odd integers for n = 1, 2, 3, 4, 5, 6 are:
      1 = **1** 1 + 3 = **4** 1 + 3 + 5 = **9**
      1 + 3 + 5 + 7 = **16** 1 + 3 + 5 + 7 + 9 = **25** 1 + 3 + 5 + 7 + 9 + 11 = **36**
    - Are you seeing a trend? Is the formula jumping out at you?
    - You should notice that the sum of the first n positive odd integers is n^2.
    - We now need to prove that this conjecture is true.
    - We do so via Induction. And by remembering the definition of odd integers, we can write this in summation form…

Mathematical Induction

- Prove the following via Induction:

$$\forall n\, P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i - 1 = n^2$$

1) **<u>Base Case:</u>** *Show that P(1) is true*. We want to confirm that the left side (LS) equals the right side (RS) when we plug in the smallest value (1) .

   - LS: $P(1) = \sum_{i=1}^{1} 2i - 1 = 1$ RS: $1^2 = 1$

Mathematical Induction

2) **<u>Inductive hypothesis:</u>** assume true for *k*

- Thus, we assume that P(*k*) is true, or that

$$\sum_{\boxed{k}}^{} \sum_{i=1}^{k} 2i - 1 == \boxed{k}^2$$

Assume P(k) is True

- Note: we do <u>not</u> know if this is true or not!
- BUT, we **<u>assume</u>** it is true!

3) **<u>Inductive step:</u>** show true for *k*+1

- We want to show that:

$$\sum_{i=1}^{\boxed{k+1}} 2i - 1 == \boxed{(k+1)}^2$$

Mathematical Induction

- Prove the following: $\sum_{i=1}^{n} 2i-1 = n^2$ by Induction
  - Remember, we need to **prove** the Inductive Step:

    $$\sum_{i=1}^{k+1} 2i-1 == (k+1)^2$$

    PROVE This!
  - Again, we use "the thing" from earlier

    $$\sum_{i=1}^{k+1} 2i-1 = \left(\sum_{i=1}^{k} 2i-1\right) + 2(k+1)-1$$
  - Why is the above correct? Because summing to the (k+1)st term is the same as summing to the kth term and then adding on the last k+1 term.
    - And this is what we did here.

Mathematical Induction

- Prove the following: $\sum_{i=1}^{n} 2i-1 = n^2$ by Induction
  - Remember, we need to **prove** the Inductive Step:

    $$\sum_{i=1}^{k+1} 2i-1 == (k+1)^2$$

    PROVE This!

  - Again, we use "the thing" from earlier

    $$\sum_{i=1}^{k+1} 2i-1 = \left(\sum_{i=1}^{k} 2i-1\right) + 2(k+1)-1$$

  - Substitute this in…

Guess what:
From our Induction Hypothesis, we know that $\sum_{i=1}^{k} 2i-1 = k^2$

Mathematical Induction

- Prove the following: $\sum_{i=1}^{n} 2i-1 = n^2$ by Induction
  - Remember, we need to **prove** the Inductive Step:

$$\boxed{\sum_{i=1}^{k+1} 2i-1 == (k+1)^2} \Leftarrow$$

PROVE This!

  - Again, we use "the thing" from earlier

$$\sum_{i=1}^{k+1} 2i-1 = \underbrace{\left(\sum_{i=1}^{k} 2i-1\right)} + 2(k+1)-1$$

Guess what:
From our Induction Hypothesis,
we know that $\sum_{i=1}^{k} 2i-1 = k^2$

$$\boxed{\sum_{i=1}^{k+1} 2i-1} = k^2 + 2(k+1)-1 = k^2 + 2k+1 = \boxed{(k+1)^2}$$

  - So we successfully used the I.H. to help us solve the I.S.

Mathematical Induction

- Prove the following: $\sum_{i=1}^{n} 2i-1 = n^2$ by Induction
  - Remember, we need to **prove** the Inductive Step:
    $$\sum_{i=1}^{k+1} 2i-1 == (k+1)^2$$
  - We just proved the Induction Step above. We proved that P(k+1) is true under the assumption that P(k) is true.
  - Since we completed all three steps, by Mathematical Induction, we know that P(n) is true for all positive integers n.
  - That is, we have proven, for all positive n, that
    $$\sum_{i=1}^{n} 2i-1 = n^2$$

Mathematical Induction

- Example #3: Prove via Mathematical Induction:

$$\sum_{i=1}^{n} i2^{i} = (n-1)2^{n+1} + 2$$

1) **<u>Base Case:</u>** *Show that P(1) is true*. We want to confirm that the left side (LS) equals the right side (RS) when we plug in the smallest value (1).
   - LS: $\sum_{i=1}^{1} i2^{i} = 1 * 2^{1} = 2$ RS: $(1-1) * 2^{1+1} + 2 = 0 + 2 = 2$

2) **<u>Induction Hypothesis:</u>** assume true for an arbitrary k

$$\sum_{i=1}^{k} i2^{i} = (k-1)2^{k+1} + 2$$

⟸ **Assume P(k) is True**

Mathematical Induction

- Example #3: Prove $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$ via Induction

3) **<u>Induction Step:</u>** based on the I.H., prove P(k+1).

***At this step, you basically plug in a k+1 wherever you see a k.*

Prove $\sum_{i=1}^{k+1} i2^i = (k+1-1)2^{k+1+1} + 2$

Prove $\sum_{i=1}^{k+1} i2^i = (k)2^{k+2} + 2$ **PROVE This!**

*All we did was simplify a bit.

Again, to prove this, we use "the thing" from earlier, which basically allows us to then use the I.H.

$$\sum_{i=1}^{k+1} i2^i = \left(\sum_{i=1}^{k} i2^i\right) + (k+1)2^{k+1}$$

Guess what:
From our Induction Hypothesis, we know that $\sum_{i=1}^{k} i2^i = (k-1)2^{k+1} + 2$

So we substitute this in…

Mathematical Induction

- Example #3: Prove $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$ via Induction

3) **<u>Induction Step:</u>** based on the I.H., prove P(k+1).

Prove $\sum_{i=1}^{k+1} i2^i = (k)2^{k+2} + 2$ **PROVE This!**

Again, to prove this, we use "the thing" from earlier, which basically allows us to then use the I.H.

$$\sum_{i=1}^{k+1} i2^i = \underbrace{\left(\sum_{i=1}^{k} i2^i\right)} + (k+1)2^{k+1}$$

$$\sum_{i=1}^{k+1} i2^i = \overbrace{(k-1)2^{k+1} + 2} + (k+1)2^{k+1}$$

Guess what:
From our Induction Hypothesis, we know that $\sum_{i=1}^{k} i2^i = (k-1)2^{k+1} + 2$

Mathematical Induction

- Example #3: Prove $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$ via Induction

3) **Induction Step:** based on the I.H., prove P(k+1).

Prove $\sum_{i=1}^{k+1} i2^i = (k)2^{k+2} + 2$ ⇦ **PROVE This!**

Again, to prove this, we use "the thing" from earlier, which basically allows us to then use the I.H.

$$\sum_{i=1}^{k+1} i2^i = \underbrace{\left(\sum_{i=1}^{k} i2^i\right)} + (k+1)2^{k+1}$$

$$\sum_{i=1}^{k+1} i2^i = \overbrace{(k-1)2^{k+1} + 2} + (k+1)2^{k+1}$$

Guess what:
From our Induction Hypothesis, we know that $\sum_{i=1}^{k} i2^i = (k-1)2^{k+1} + 2$

Mathematical Induction

- Example #3: Prove $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$ via Induction

3) **Induction Step:** based on the I.H., prove P(k+1).

Prove $\boxed{\sum_{i=1}^{k+1} i2^i = (k)2^{k+2} + 2}$ **PROVE This!**

$$\sum_{i=1}^{k+1} i2^i = (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$

The rest is just basic Algebra

$$\sum_{i=1}^{k+1} i2^i = (k-1)2^{k+1} + (k+1)2^{k+1} + 2 = 2^{k+1}(2k) + 2$$

*We re-ordered and then factored out a 2^{k+1}.

$$\boxed{\sum_{i=1}^{k+1} i2^i} = 2^{k+1}(2k) + 2 = (k)2^1 2^{k+1} + 2 = \boxed{(k)2^{k+2} + 2}$$

- Again, we successfully used the I.H. to help us solve the I.S.

Mathematical Induction

- Example #3: Prove $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$ via Induction
  - Remember, we need to **prove** the Inductive Step:

    $$\sum_{i=1}^{k+1} i2^i = (k)2^{k+2} + 2$$

  - We just proved the Induction Step above. We proved that P(k+1) is true under the assumption that P(k) is true.
  - Since we completed all three steps, by Mathematical Induction, we know that P(n) is true for all positive integers n.
  - That is, we have proven, for all positive n, that

    $$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$$

Today’s demotivators

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Mathematical Induction

WASN'T

THAT

MOTIVATIONAL!

Mathematical Induction

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