Counting

College of Computing & Information Technology
King Abdulaziz University

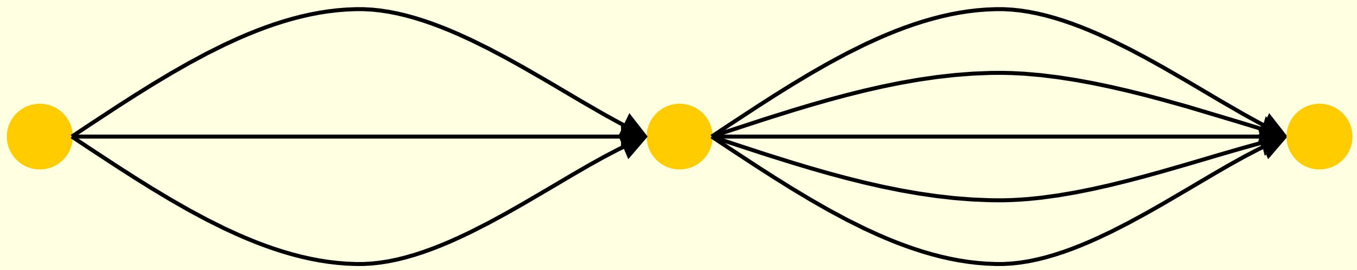
CPCS-222 – Discrete Structures

Basics of Counting

- Section 5.1

The Product Rule

- If there are n_1 ways to do task 1, and n_2 ways to do task 2
  - Then there are $n_1 n_2$ ways to do both tasks in sequence
  - This applies when doing the “procedure” is made up of separate tasks
  - We must make one choice AND a second choice



Product rule example

- Rosen, section 5.1, question 1 (a)
  - There are 18 math majors and 325 CS majors
  - How many ways are there to pick one math major and one CS major?
- Total is 18 * 325 = 5850

Product rule example

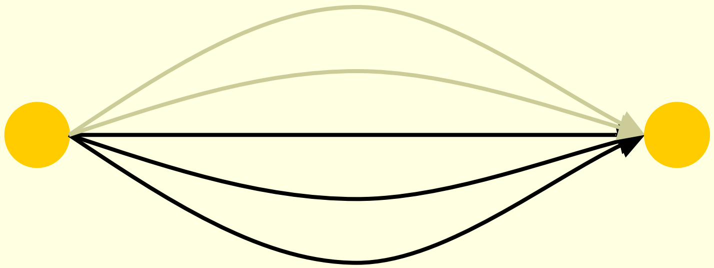
- Rosen, section 5.1, question 24 (a) and (b)
- How many strings of 4 decimal digits…

a) Do not contain the same digit twice?
   - We want to chose a digit, then another that is not the same, then another…
     - First digit: 10 possibilities
     - Second digit: 9 possibilities (all but first digit)
     - Third digit: 8 possibilities
     - Fourth digit: 7 possibilities
   - Total = 10*9*8*7 = 5040

b) End with an even digit?
   - First three digits have 10 possibilities
   - Last digit has 5 possibilities
   - Total = 10*10*10*5 = 5000

The Sum Rule

- If there are n_1 ways to do task 1, and n_2 ways to do task 2
  - If these tasks can be done at the same time, then…
  - Then there are n_1+n_2 ways to do one of the two tasks
  - We must make one choice OR a second choice



Sum rule example

- Rosen, section 5.1, question 1 (b)
  - There are 18 math majors and 325 CS majors
  - How many ways are there to pick one math major or one CS major?
- Total is 18 + 325 = 343

Sum rule example

- Rosen, section 5.1, question 24 (c)
- How many strings of 4 decimal digits…

c) Have exactly three digits that are 9s?
  - The string can have:
    - The non-9 as the first digit
    - OR the non-9 as the second digit
    - OR the non-9 as the third digit
    - OR the non-9 as the fourth digit
    - Thus, we use the sum rule
  - For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
  - Thus, the answer is 9+9+9+9 = 36

More complex counting problems

- We combine the product rule and the sum rule

- Thus we can solve more interesting and complex problems

Wedding pictures example

- Rosen, section 5.1, question 40
- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

a) How many possibilities are there if the bride must be in the picture

- Product rule: place the bride AND then place the rest of the party
- First place the bride
  - She can be in one of 6 positions
- Next, place the other five people via the product rule
  - There are 9 people to choose for the second person, 8 for the third, etc.
  - Total = 9*8*7*6*5 = 15120
- Product rule yields 6 * 15120 = 90,720 possibilities

Wedding pictures example

- Rosen, section 5.1, question 40
- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

b) How many possibilities are there if the bride and groom must both be in the picture
  - Product rule: place the bride/groom AND then place the rest of the party
  - First place the bride and groom
    - She can be in one of 6 positions
    - He can be in one 5 remaining positions
    - Total of 30 possibilities
  - Next, place the other four people via the product rule
    - There are 8 people to choose for the third person, 7 for the fourth, etc.
    - Total = 8*7*6*5 = 1680
  - Product rule yields 30 * 1680 = 50,400 possibilities

Wedding pictures example

- Rosen, section 5.1, question 40
- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

c) How many possibilities are there if only one of the bride and groom are in the picture
  - Sum rule: place only the bride
    - Product rule: place the bride AND then place the rest of the party
    - First place the bride
      - She can be in one of 6 positions
    - Next, place the other five people via the product rule
      - There are 8 people to choose for the second person, 7 for the third, etc.
        - We can't choose the groom!
      - Total = 8*7*6*5*4 = 6720
    - Product rule yields 6 * 6720 = 40,320 possibilities
  - OR place only the groom
    - Same possibilities as for bride: 40,320
  - Sum rule yields 40,320 + 40,320 = 80,640 possibilities

Wedding pictures example

- Rosen, section 5.1, question 40
- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

- Alternative means to get the answer

c) How many possibilities are there if only one of the bride and groom are in the picture
  - Total ways to place the bride (with or without groom): 90,720
    - From part (a)
  - Total ways for both the bride and groom: 50,400
    - From part (b)
  - Total ways to place ONLY the bride: 90,720 – 50,400 = 40,320
  - Same number for the groom
  - Total = 40,320 + 40,320 = 80,640

Brief Interlude: Human Stupidity

Daily Bike Fail

The inclusion-exclusion principle

- When counting the possibilities, we can’t include a given outcome more than once!

- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - Let A_1 have 5 elements, A_2 have 3 elements, and 1 element be both in A_1 and A_2
  - Total in the union is 5+3-1 = 7, not 8

Inclusion-exclusion example

- Rosen, section 5.1, example 17
- How may bit strings of length eight start with 1 or end with 00?

- Count bit strings that start with 1
  - Rest of bits can be anything: $2^7 = 128$
  - This is $|A_1|$
- Count bit strings that end with 00
  - Rest of bits can be anything: $2^6 = 64$
  - This is $|A_2|$
- Count bit strings that both start with 1 and end with 00
  - Rest of the bits can be anything: $2^5 = 32$
  - This is This is $|A_1 \cap A_2|$

- Use formula $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- Total is $128 + 64 – 32 = 160$

Bit string possibilities

- Rosen, section 5.1, question 44
- How many bit strings of length 10 contain either 5 consecutive 0s or 5 consecutive 1s?

Bit string possibilities

- How many bit strings of length 10 contain either 5 consecutive 0s or 5 consecutive 1s?
- Consider 5 consecutive 0s first
- Sum rule: the 5 consecutive 0's can start at position 1, 2, 3, 4, 5, or 6
  - Starting at position 1
    - Remaining 5 bits can be anything: $2^5 = 32$
  - Starting at position 2
    - First bit must be a 1
      - Otherwise, we are including possibilities from the previous case!
    - Remaining bits can be anything: $2^4 = 16$
  - Starting at position 3
    - Second bit must be a 1 (same reason as above)
    - First bit and last 3 bits can be anything: $2^4 = 16$
  - Starting at positions 4 and 5 and 6
    - Same as starting at positions 2 or 3: 16 each
  - Total = 32 + 16 + 16 + 16 + 16 + 16 = 112

Bit string possibilities

- How many bit strings of length 10 contain either 5 consecutive 0s or 5 consecutive 1s?
- Now consider 5 consecutive 1s first
- The 5 consecutive 1’s follow the same pattern, and have 112 possibilities
- There are two cases counted twice (that we thus need to exclude): 0000011111 and 1111100000
- Total = 112 + 112 – 2 = 222

The Pigeonhole Principle

- Section 5.2

The pigeonhole principle

- Suppose a flock of pigeons fly into a set of pigeonholes to roost
- If there are more pigeons than pigeonholes, then there must be at least 1 pigeonhole that has more than one pigeon in it
- If k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects
  - This is Theorem 1

Pigeonhole principle examples

- In a group of 367 people, there must be two people with the same birthday
  - As there are 366 possible birthdays
- In a group of 27 English words, at least two words must start with the same letter
  - As there are only 26 letters

Generalized pigeonhole principle

- If N objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ objects
  - This is Theorem 2

Generalized pigeonhole principle examples

- Among 100 people, there are at least $\lceil 100/12 \rceil = 9$ born on the same month

- How many students in a class must there be to ensure that 6 students get the same grade (one of A, B, C, D, or F)?
  - The "boxes" are the grades. Thus, $k = 5$
  - Thus, we set $\lceil N/5 \rceil = 6$
  - Lowest possible value for N is 26

Rosen, section 5.2, question 4

- A bowl contains 10 red and 10 yellow balls

a) How many balls must be selected to ensure 3 balls of the same color?
   - One solution: consider the "worst" case
     - Consider 2 balls of each color
     - You can't take another ball without hitting 3
     - Thus, the answer is 5
   - Via generalized pigeonhole principle
     - How many balls are required if there are 2 colors, and one color must have 3 balls?
     - How many pigeons are required if there are 2 pigeon holes, and one must have 3 pigeons?
     - number of boxes: $k = 2$
     - We want $\lceil N/k \rceil = 3$
     - What is the minimum N?
     - $N = 5$

Rosen, section 5.2, question 4

- A bowl contains 10 red and 10 yellow balls

b) How many balls must be selected to ensure 3 yellow balls?

- Consider the “worst” case
  - Consider 10 red balls and 2 yellow balls
  - You can’t take another ball without hitting 3 yellow balls
  - Thus, the answer is 13

Rosen, section 5.2, question 32

- 6 computers on a network are connected to at least 1 other computer
- Show there are at least two computers that are have the same number of connections

- The number of boxes, *k*, is the number of computer connections
  - This can be 1, 2, 3, 4, or 5
- The number of pigeons, *N*, is the number of computers
  - That's 6
- By the generalized pigeonhole principle, at least one box must have $\lceil N/k \rceil$ objects
  - $\lceil 6/5 \rceil = 2$
  - In other words, at least two computers must have the same number of connections

Permutations & Combinations

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Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether order matters or not
- Consider a poker hand:
  - A♦, 5♥, 7♣, 10♠, K♠
- Is that the same hand as:
  - K♠, 10♠, 7♣, 5♥, A♦
- Does the order the cards are handed out matter?
  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations

Permutations

- A permutation is an ordered arrangement of the elements of some set *S*
  - Let *S* = {a, b, c}
  - c, b, a is a permutation of S
  - b, c, a is a *different* permutation of S
- An *r*-permutation is an ordered arrangement of *r* elements of the set
  - A♦, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards
- The notation for the number of *r*-permutations: $P(n,r)$
  - The poker hand is one of P(52,5) permutations

Permutations

- Number of poker hands (5 cards):
  - *P*(52,5) = 52*51*50*49*48 = 311,875,200
- Number of (initial) blackjack hands (2 cards):
  - *P*(52,2) = 52*51 = 2,652
- *r*-permutation notation: *P*(*n*,*r*)
  - The poker hand is one of P(52,5) permutations

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

$$\boxed{= \frac{n!}{(n-r)!}}$$

$$= \prod_{i=n-r+1}^{n} i$$

r-permutations example

- How many ways are there for 5 people in this class to give presentations?

- There are 27 students in the class
  - P(27,5) = 27*26*25*24*23 = 9,687,600
  - Note that the order they go in does matter in this example!

Permutation formula proof

- There are n ways to choose the first element
  - n-1 ways to choose the second
  - n-2 ways to choose the third
  - …
  - n-r+1 ways to choose the r^{th} element
- By the product rule, that gives us:

  $P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$

Permutations vs. *r*-permutations

- *r*-permutations: Choosing an ordered 5 card hand is $P(52,5)$
  - When people say "permutations", they almost always mean *r*-permutations
    - But the name can refer to both
- Permutations: Choosing an order for all 52 cards is $P(52,52) = 52!$
  - Thus, $P(n,n) = n!$

Rosen, section 4.3, question 3

- How many permutations of {a, b, c, d, e, f, g} end with a?
  - Note that the set has 7 elements

- The last character must be a
  - The rest can be in any order
- Thus, we want a 6-permutation on the set {b, c, d, e, f, g}
- $P(6,6) = 6! = 720$

- Why is it not $P(7,6)$?

Combinations

- What if order *doesn't* matter?
- In poker, the following two hands are equivalent:
  - A♦, 5♥, 7♣, 10♠, K♠
  - K♠, 10♠, 7♣, 5♥, A♦
- The number of *r*-combinations of a set with *n* elements, where *n* is non-negative and $0 \le r \le n$ is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Combinations example

- How many different poker hands are there (5 cards)?

$$C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{52*51*50*49*48*47!}{5*4*3*2*1*47!} = 2{,}598{,}960$$

- How many different (initial) blackjack hands are there?

$$C(52,2) = \frac{52!}{2!(52-2)!} = \frac{52!}{2!50!} = \frac{52*51}{2*1} = 1{,}326$$

Combination formula proof

- Let $C(52,5)$ be the number of ways to generate unordered poker hands
- The number of ordered poker hands is $P(52,5) = 311{,}875{,}200$
- The number of ways to order a single poker hand is $P(5,5) = 5! = 120$
- The total number of unordered poker hands is the total number of ordered hands divided by the number of ways to order each hand
- Thus, $C(52,5) = P(52,5)/P(5,5)$

Combination formula proof

- Let $C(n,r)$ be the number of ways to generate unordered combinations
- The number of ordered combinations (i.e. r-permutations) is $P(n,r)$
- The number of ways to order a single one of those r-permutations $P(r,r)$
- The total number of unordered combinations is the total number of ordered combinations (i.e. r-permutations) divided by the number of ways to order each combination
- Thus, $C(n,r) = P(n,r)/P(r,r)$

Combination formula proof

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

- Note that the textbook explains it slightly differently, but it is same proof

Rosen, section 4.3, question 11

- How many bit strings of length 10 contain:

a) exactly four 1's?
- Find the positions of the four 1's
- Does the order of these positions matter?
  - Nope!
  - Positions 2, 3, 5, 7 is the same as positions 7, 5, 3, 2
- Thus, the answer is $C(10,4) = 210$

b) at most four 1's?
- There can be 0, 1, 2, 3, or 4 occurrences of 1
- Thus, the answer is:
  - $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4)$
  - $= 1+10+45+120+210$
  - $= 386$

Rosen, section 4.3, question 11

- How many bit strings of length 10 contain:

c) **at least four 1's?**
  - There can be 4, 5, 6, 7, 8, 9, or 10 occurrences of 1
  - Thus, the answer is:
    - $C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10)$
    - $= 210+252+210+120+45+10+1$
    - $= 848$
  - Alternative answer: subtract from 2^{10} the number of strings with 0, 1, 2, or 3 occurrences of 1

d) **an equal number of 1's and 0's?**
  - Thus, there must be five 0's and five 1's
  - Find the positions of the five 1's
  - Thus, the answer is $C(10,5) = 252$

Corollary 1

- Let *n* and *r* be non-negative integers with $r \le n$. Then $C(n,r) = C(n,n-r)$

- Proof:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{r!(n-r)!}$$

Corollary example

- There are C(52,5) ways to pick a 5-card poker hand
- There are C(52,47) ways to pick a 47-card hand
- P(52,5) = 2,598,960 = P(52,47)

- When dealing 47 cards, you are picking 5 cards to not deal
  - As opposed to picking 5 card to deal
  - Again, the order the cards are dealt in does matter

Combinatorial proof

- A *combinatorial proof* is a proof that uses counting arguments to prove a theorem
  - Rather than some other method such as algebraic techniques
- Essentially, show that both sides of the proof manage to count the same objects
- In a typical Rosen example, he does not do much with this proof method in this section
  - We will see more in the next sections
- Most of the questions in this section are phrased as, “find out how many possibilities there are if …”
  - Instead, we could phrase each question as a theorem:
  - “Prove there are *x* possibilities if …”
  - The same answer could be modified to be a combinatorial proof to the theorem

Rosen, section 4.3, question 40

- How many ways are there to sit 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?
- First, place the first person in the north-most chair
  - Only one possibility
- Then place the other 5 people
  - There are P(5,5) = 5! = 120 ways to do that
- By the product rule, we get 1*120 =120

- Alternative means to answer this:
  - There are P(6,6)=720 ways to seat the 6 people around the table
  - For each seating, there are 6 “rotations” of the seating
  - Thus, the final answer is 720/6 = 120

Rosen, section 4.3, question 42

- How many ways are there for 4 horses to finish if ties are allowed?
  - Note that order does matter!
- Solution by cases
  - No ties
    - The number of permutations is P(4,4) = 4! = 24
  - Two horses tie
    - There are *C*(4,2) = 6 ways to choose the two horses that tie
    - There are *P*(3,3) = 6 ways for the “groups” to finish
      - A “group” is either a single horse or the two tying horses
    - By the product rule, there are 6*6 = 36 possibilities for this case
  - Two groups of two horses tie
    - There are *C*(4,2) = 6 ways to choose the two winning horses
    - The other two horses tie for second place
  - Three horses tie with each other
    - There are *C*(4,3) = 4 ways to choose the two horses that tie
    - There are *P*(2,2) = 2 ways for the “groups” to finish
    - By the product rule, there are 4*2 = 8 possibilities for this case
  - All four horses tie
    - There is only one combination for this
  - By the sum rule, the total is 24+36+6+8+1 = 75

Permutations & Combinatios

WASN'T THAT BODACIOUS

Today's demotivators

Counting

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